International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 6, Oct – Nov 2017; 1-8

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

INVARIANCE PROPERTIES AND ESTIMATING TASK SOLUTION OF BIOLOGICAL POPULATION IN THE TWO-DIMENSIONAL CASE

MUHAMEDIYEVA D. K.

PhD in Technical Sciences, Research and Innovation Center in Information and Communication Technologies at the Tashkent University of Information Technology Named After Muhammad Al-Kharezmi, Tashkent, Uzbekistan

ABSTRACT

Considered methods of solving Kolmogorov - Fisher type task of reacting with diffusion, in the two-dimensional case, obtained invariant properties of solutions and two way estimation of the solution. Given are the results of numerical experiments for various values included in the equation parameters, in the two-dimensional case.

KEYWORDS: Invariance properties, nonlinear problems, task solution, biological population

UDC: 577.3.01; 577.38

1. INTRODUCTION

Currently, studies are on various properties of solutions of nonlinear problems in large number of works, and identifying all new and newer properties of their solutions. Solution of nonlinear boundary value problems are always accompanied by considerable difficulties, because, to solve them in an analytical form is possible, only in exceptional cases and for establishing new properties of the solutions require painstaking research. Standard methods of research on nonlinear problems, depending on the type of nonlinearities are not yet available. Therefore, in each case, to study the properties of solutions have resorted to various exact and approximate methods, in leading universities and research centers around the world, including Oxford University, University of Cambridge, Department of Mathematical Sciences of University of Liverpool, Department of Applied Mathematics of the University of Leeds Faculty of Biological Sciences of the University of Leeds, Department of Engineering Mathematics and School of Biological Sciences of University of Bristol (UK), Department of Mathematics of Stanford University, Consortium of the Americas for Interdisciplinary Science and Department of Physics and Astronomy of University of New Mexico, Department of Biology of University of Louisiana, Departments of Entomology and Biology, Pennsylvania State University (USA), Ecole des Hautes Etudes en Sciences Sociales, Institut de Mathématiques de Toulouse, Universite Paul Sabatier (France), Universit'a degli Studi di Padova Dipartimento di Matematica (Italy), Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile Complex Systems Group, Facultad de Ingeniería y Ciencias Aplicadas, Universidad de los Andes (Chile), Centro Ato mico Bariloche, Instituto Balseiro and CONICET (Argentina), Department of Theoretical Ecology, Biology Centre all ASCR, Institute of Entomology (Czech Republic), School of Sciences, Jimei University, Xiamen (China), Moscow State University, Institute of applied mathematics, Institute of theoretical and experimental Biophysics, Tomsk State University (Russia), national University of Uzbekistan, Samarkand state University (Uzbekistan).

Review of international research, based on the works of Oxford University, University of Cambridge, Department of Mathematical Sciences of University of Liverpool, Department of Engineering, Mathematics and School of Biological Sciences of the University of Bristol, Department of Mathematics of Stanford University, Consortium of the Americas for

Interdisciplinary Science and Department of Physics and Astronomy of the University of New Mexico [1-7].

Review of foreign scientific research shows that, since the work of Turing, the mathematical model of the formation and propagation of nonlinear waves and processes of a structured self - organization in physical, chemical, biological, and social systems such as reaction-diffusion, where members describe nonlinear kinetics, and transport processes, represented by isotropic diffusion (Oxford University, University of Cambridge). However, in many systems, no less important, more complex mechanisms, diffusion is a nonlinear, anisotropic and cross-diffusion. Most studies have examined direct diffusion (self-diffusion) described by the equations, the diffusion coefficients, which are constant values (Ecole des Hautes Etudes en Sciences Sociales, Institut de Mathématiques de Toulouse, Universite Paul Sabatier (France), Universit'a degli Studi di Padova Dipartimento di Matematica (Italy)). In this class of systems, the processes of formation of spatial-temporal structures are determined by the diffusion coefficients and the specific forms of the kinetic function of the reaction process. The case when the diffusion coefficients are not constants, but depend on dynamic variables, corresponds to the nonlinear diffusion. Examples of nonlinear diffusion, in the processes of mass transfer in porous environment, as well as in population models (Consortium of the Americas for Interdisciplinary Science and Department of Physics and Astronomy of the University of New Mexico). Mathematical models with diffusion coefficient depending on the density of bacteria, describes the processes of formation of complex spatial structures with the growth of bacterial colonies. Regimes with aggravation in the spatial localization, in open dissipative systems, described by models with nonlinear diffusion (Institute of applied mathematics, Institute of theoretical and experimental Biophysics, Tomsk State University).

2. STATEMENT OF THE TASK

Consider a two dimensional problem of Kolmogorov - Fisher type for the reaction diffusion

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_1}\left[u^{\sigma}\frac{\partial}{\partial x_1}u\right]+\frac{\partial}{\partial x_2}\left[u^{\sigma}\frac{\partial}{\partial x_2}u\right]+k(t,x)u\left(1-u^{\beta}\right) \tag{1}$$
$$u=u(x_1,x_2,t),\quad |x|=\sqrt{(x_1)^2+(x_2)^2}$$

In the domain $D=\Omega\times(0,T),\ \Omega\subset R^2,\ \ \Omega=\{-b_\alpha<x_\alpha<b_\alpha,\ \alpha=1,2\}$ with the initial and boundary conditions

$$u(0,x)=u_0\left(x\right)\geq 0, \tag{2}$$

$u\big|_{\Gamma}=\mu(x,t)$, $t\in(0,T)$, Γ – boundary of Ω.

To solve this problem (1), (2) uses the initial approximation

$$u_0(t,x)=\psi(t)\left(a-\frac{\sigma}{4}\xi^2\right)_+^{1/\sigma};\ \ a=1;\ \ \xi=|x|/\tau^{1/2};\quad \tau(t)=\int_0^t[\psi(\eta)]^{\sigma}d\eta,$$

in the case $k(t,x):=k(t)$; $k(t)=\dfrac{1}{(1+t)^{\alpha}}$, $\alpha>1$; $\psi(t)=\left(1+e^{-\beta\int_0^t k(t)dt}\right)^{-\frac{1}{\beta}}$.

In this case, obtained the following condition for the localization of solutions of problem (1), (2): $\tau(\infty) < +\infty$, $q\int e^{\int_0^t k(\eta)d\eta} d\eta < +\infty$.

Note that, the way to build initial approximation based on the splitting of the equation (1) [1].

This task in the case *m=0 и k(t,x)=const* was studied in several papers [2,3-5].

In [6] derived the invariant properties of solutions. As in [2] for the models of Malthus, Ferhulst, Olli obtained properties of the finite speed of the perturbations. In [7] established properties of flash localization.

PROPERTIES OF INVARIANCE OF THE SOLUTION

Let's show that the function

$$u(t,x) = \psi(t) w(\tau(t), x), \tag{3}$$

where $\psi(t)$ solution to the equation without diffusive part of equation (1)

$$\frac{d\psi}{dt} = k(t)\psi(1-\psi^{\beta}) \text{, т.е. } \psi(t) = \left(1 + e^{-\beta\int_0^t k(t)dt}\right)^{-\frac{1}{\beta}}, \tag{4}$$

again satisfies the equation (1).

In fact, after putting (3) into (1) subject to (3), (4) it is easy to calculate that for $w(t, x)$ have the equation

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x_1}\left[w^{\sigma}\frac{\partial}{\partial x_1}w\right] + \frac{\partial}{\partial x_2}\left[w^{\sigma}\frac{\partial}{\partial x_2}w\right] + \psi_1(t)w(1-w^{\beta}),$$
$$w = w(x_1, x_2, \tau(t)), \quad |x| = \sqrt{(x_1)^2 + (x_2)^2},$$

where $w(\tau(t), x)$ new unknown function,a $\tau(t)$ - to be determined function.

$\psi(t)$ we find from the equation:

$$\frac{d\psi}{dt} = k(t)\psi(1-\psi^{\beta}), \ \psi(t) = \left(1 + e^{-\beta\int_0^t k(t)dt}\right)^{-\frac{1}{\beta}}.$$

Substituting (3) into (1) and choosing $\tau(t)$ from $\frac{d\tau}{dt} = \psi^{\sigma}$ have:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x_1}\left[w^{\sigma}\frac{\partial}{\partial x_1}w\right] + \frac{\partial}{\partial x_2}\left[w^{\sigma}\frac{\partial}{\partial x_2}w\right] + k(t)\psi^{\beta-\sigma}w(1-w^{\beta}), \tag{5}$$

where $\psi_1(t) = k(t)\psi^{\beta-\sigma}$.

However, it is clear that due to (4) $\lim\limits_{t\to\infty}\psi(t) = 1$, if $\int_0^t k(t)dt$ exist. Therefore, we can assume that for sufficiently large t, $\psi_1(t) \sim k(t)$. ie will again get the equation (1). Because of this, we call the function $\psi(t)w(\tau(t),\, x)$ where $\psi(t)$- the solution of equation (4), and $w(\tau(t),\, x)$ solution of equation (5) invariant of the equation (1).

For the solution of the problem (1), (2) fair

Theorem 1

Let $0 \le u_0(x) \le 1,\ x \in R,\ \xi = |x|/\tau^{1/2}$.

Then for the solution of the task (1), (2) in $Q = \{(t,\ x);\ t > 0,\ x \in R^N\}$ takes place two-way estimation

$$\psi(t)(T+\tau(t))^{-N/(2+\sigma N)}(a-\frac{\sigma}{4}\xi^2)_+^{\frac{1}{\sigma}} \le u(t,x) \le e^{\int_0^t k(\eta)\psi^{\beta-\sigma}(\eta)d\eta}(T+\tau(t))^{-N/(2+\sigma N)}(a-\frac{\sigma}{4}\xi^2)_+^{\frac{1}{\sigma}},$$

where $\psi(t)$ defined above function, $N=2$.

Proof

For the proof of the theorem first get the estimate from above. To this end, in (1) will replace

$$u(t,x) = e^{\int_0^t k(\eta)d\eta}w(\tau(t),\, x). \quad \textbf{(6)}$$

Then for $w(\tau(t),x)$ have the equation

$$\frac{\partial w}{\partial \tau} = \nabla(w^\sigma \nabla w) - k(t)e^{-\beta\int_0^t k(\eta)d\eta}w^\beta. \quad \textbf{(7)}$$

Function $w_+(\tau(t),x) = (T+\tau(t))^{-N/(2+\sigma N)}(a-\frac{\sigma}{4}\xi^2)_+^{\frac{1}{\sigma}}$ is the upper solution of equation (7) as $w_+(\tau(t),x)$ is a solution of the equation $\frac{\partial w}{\partial \tau} = \nabla(w^\sigma \nabla w)$ and $-k(t)e^{-\beta\int_0^t k(\eta)d\eta}w^\beta \le 0$ in Q for any constant T>0.

Therefore, according to comparison theorem of solutions [1] has an upper bound

$$u(t,x) \le e^{\int_0^t k(\eta)\psi^{\beta-\sigma}(\eta)d\eta}w_+(\tau(t),x) \quad \textbf{(8)}$$

in Q, if $w_+(0,\ x) \le u_0(x), x \in R^N$.

In order to obtain a lower bound, we apply the method of nonlinear splitting [1]. According to this method the bottom the solution is sought in the form

$$u(t,x)=\psi(t)w_-\left(\tau(t),x\right), \tag{9}$$

Where $\psi(t)$ above-defined by formula (4) function.

Then, from (5) we have

$$\frac{\partial w_-}{\partial \tau}=\frac{\partial}{\partial x_1}\left[w_-^{\ \sigma}\frac{\partial}{\partial x_1}w\right]+\frac{\partial}{\partial x_2}\left[w_-^{\ \sigma}\frac{\partial}{\partial x_2}w\right]+k(t)\psi^{\beta-\sigma}w_-\left(1-w_-^{\ \beta}\right). \tag{10}$$

For function $(T+\tau(t))^{-N/(2+\sigma N)}(a-\frac{\sigma}{4}\xi^2)_+^{\frac{1}{\sigma}}$

$k(t)\psi^{\beta-\sigma}w_-\left(1-w_-^{\ \beta}\right)\geq 0,$ in Q, if constant $T\geq 1$.

Then applying the comparison theorem of solutions [1] because of (10) have

$$u(t,x)\geq\psi(t)(T+\tau(t))^{-N/(2+\sigma N)}(a-\frac{\sigma}{4}\xi^2)_+^{\frac{1}{\sigma}},$$

with (9) proves the validity of theorem 1.

It is an important generalization of this task on different occasions. In particular, it has not been studied in the case of heterogeneous medium (the diffusion coefficient is a function of the spatial variable), the reaction coefficient depends on time or has a more complex character.

$$\begin{gathered}\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_1}\left[|x|^m u^\sigma\frac{\partial}{\partial x_1}u\right]+\frac{\partial}{\partial x_2}\left[|x|^m u^\sigma\frac{\partial}{\partial x_2}u\right]+k(t,x)u\left(1-u^\beta\right),\\ u=u\left(x_1,x_2,t\right),\quad |x|=\sqrt{\left(x_1\right)^2+\left(x_2\right)^2}\end{gathered} \tag{11}$$

In the domain $D=\Omega\times(0,T),\ \Omega\subset R^N,\ \ \Omega=\{-b_\alpha<x_\alpha<b_\alpha,\ \ \alpha=1{,}2\}$ with the initial and boundary conditions

$$u(0,x)=u_0\left(x\right)\geq 0, \tag{12}$$

$u\big|_\Gamma=\mu(x,t),\ t\in(0,T)$, Γ boundary of Ω.

To solve this task (11), (12) uses the initial approximation

$$u_0(t,x)=\psi(t)\left(a-\frac{\sigma}{4}\xi^2\right)_+^{1/\sigma};\ \ a=1;\ \ \ \xi=|x|/\tau^{1/2};\ \ \ \ \tau(t)=\int_0^t[\psi(\eta)]^\sigma d\eta,$$

in case $k(t,x) := k(t)$; $k(t) = \dfrac{1}{(1+t)^{\alpha}}$, $\alpha > 1$; $\psi(t) = \left(1 + e^{-\beta\int_0^t k(t)dt}\right)^{-\frac{1}{\beta}}$.

In this case, obtained the following condition for the localization of solutions of task (11), (12): $\tau(\infty) < +\infty$, $q\int e^{\int_0^t k(\eta)d\eta} d\eta < +\infty$.

Note that, the way to build initial approximation based on the splitting of equation (11) after bringing it to the radially symmetric form [1].

Show that the function

$$u(t,x) = \psi(t) w(\tau(t), x), \quad \textbf{(13)}$$

where $\psi(t)$ solution to the equation without diffusive part of equation (11)

$$\frac{d\psi}{dt} = k(t)\psi(1-\psi^{\beta}) \text{, т.е. } \psi(t) = \left(1 + e^{-\beta\int_0^t k(t)dt}\right)^{-\frac{1}{\beta}} \quad \textbf{(14)}$$

Again, it satisfies the equation (11).

In fact, after putting (13) into (11) with (13), (14) it is easy to calculate that for $w(t,\ x)$ have the equation

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x_1}\left[|x|^m w^{\sigma} \frac{\partial}{\partial x_1} w\right] + \frac{\partial}{\partial x_2}\left[|x|^m w^{\sigma} \frac{\partial}{\partial x_2} w\right] + \psi_1(t) w(1 - w^{\beta}),$$
$$w = w(x_1, x_2, \tau(t)), \quad |x| = \sqrt{(x_1)^2 + (x_2)^2},$$

where, $w(\tau(t), x)$ new unknown function, a $\tau(t)$ - to be determined function.

$\psi(t)$ we find from the equation:

$$\frac{d\psi}{dt} = k(t)\psi(1-\psi^{\beta}), \ \psi(t) = \left(1 + e^{-\beta\int_0^t k(t)dt}\right)^{-\frac{1}{\beta}}.$$

Substituting (13) into (11) and choosing $\tau(t)$ from $\dfrac{d\tau}{dt} = \psi^{\delta}$ we get:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x_1}\left[|x|^m w^{\sigma} \frac{\partial}{\partial x_1} w\right] + \frac{\partial}{\partial x_2}\left[|x|^m w^{\sigma} \frac{\partial}{\partial x_2} w\right] + k(t)\psi^{\beta-\sigma} w(1 - w^{\beta}), \quad \textbf{(15)}$$

where $\psi_1(t) = k(t)\psi^{\beta-\sigma}$.

However, it is clear that due (14) $\lim_{t\to\infty}\psi(t) = 1$, if $\int_0^t k(t)dt$ exist. Therefore, we can assume that, for sufficiently large t, $\psi_1(t) \sim k(t)$. i.e again obtain equation (11). Because of this, we call the function $\psi(t)w(\tau(t), x)$ where $\psi(t)$- solution of equation (14), and $w(\tau(t), x)$ solution of equation (15) invariant of the equation (11).

Below we study the effect of heterogeneity in *m=-2(N-1),* $N \ge 2$, where *N* dimension of the space in the following cases: $\beta < \sigma + 1$, $\beta = \sigma + 1$ and $\beta > \sigma + 1$.

Simulated picture of the evolution process is on the computer by using Matlab.

4. NUMERICAL EXPERIMENT

The following are the results of numerical experiments for various values included in the equation parameters in the two-dimensional case:

$$t \in [0,1];\ x_1 \in [-6,6];\ x_2 \in [-6,6]; \qquad n = 1.1; \qquad n_{x_1} = 30;\ n_{x_2} = 30;\ n_t = 100.$$

In all considered cases the proposed approach number of iterations on average did not exceed three at a given precision eps.

Created on input language Matlab the program allows you to visually trace the evolution process for different values of the parameters and the data (Fig. [1-4]).

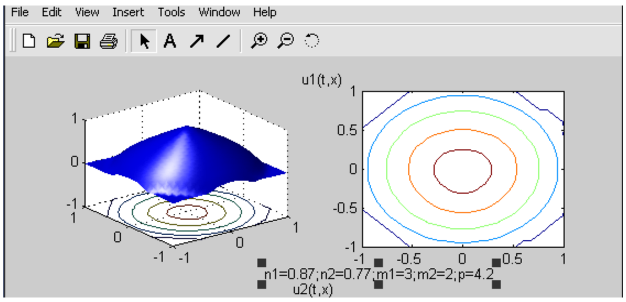

Figure 1: Results of Numerical Experiments at: $x_1 = 1; x_2 = 1; \sigma = 1; m = 0.77$

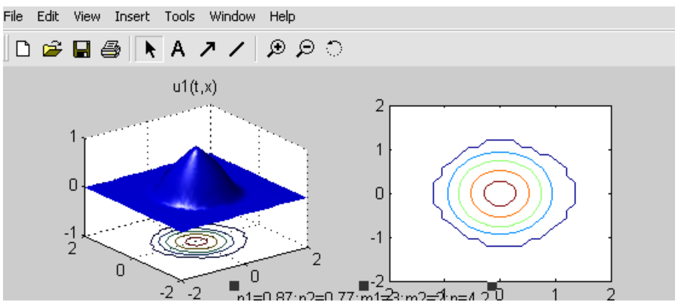

Figure 2: Results of Numerical Experiments at: $x_1 = 2; x_2 = 2; \sigma = 2; m = 0.87$

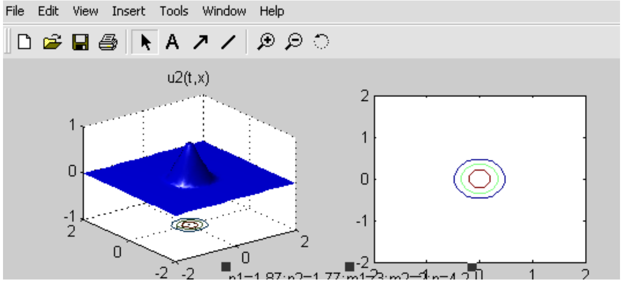

Figure 3: Results of Numerical Experiments at: $x_1 = 1; x_2 = 1;\ \sigma = 1; m = 1.77$

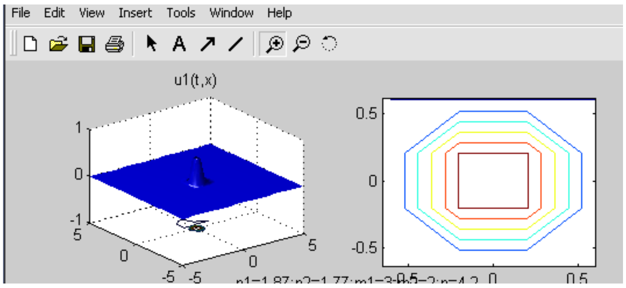

Figure4: Results of Numerical Experiments at: $x_1 = 2; x_2 = 2;\ \sigma = 2; m = 1.87$

5. CONCLUSIONS

The data of numerous experiments show that the application of computer technology to the problems of populations for a wide range of changes in input parameters allows to obtain results with a high degree of accuracy and with little expenditure of computer time.

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